

# CONJUGACY CLASSES IN KAC-MOODY GROUPS AND PRINCIPAL $G$ -BUNDLES OVER ELLIPTIC CURVES

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ABSTRACT. For a simple complex Lie group  $G$  the connected components of the moduli space of  $G$ -bundles over an elliptic curve are weighted projective spaces. In this note we will provide a new proof of this result using the invariant theory of Kac-Moody groups, in particular the action of the (twisted) Coxeter element on the root system of  $G$ .

## 1. INTRODUCTION

Let  $G$  be a simple algebraic group over  $\mathbb{C}$ . It is known ([L, FM2]) that the connected components of the moduli space of semistable  $G$ -bundles over an elliptic curve are isomorphic to weighted projective spaces. In this note, we show how this result can be obtained from the geometry of the holomorphic Kac-Moody group  $\tilde{G}$  associated to  $G$ . The main step is a detailed description of the action of the (twisted) Coxeter element of the Weyl group of  $\tilde{G}$  on the root system of  $G$ . In particular, we show that the holomorphic principal bundle associated to the Coxeter element is minimally unstable in the sense of [FM2].

Let us briefly sketch the main idea assuming that  $G$  is simply connected. Let  $L(G)$  denote the holomorphic loop group of  $G$ , i.e. the group of holomorphic maps from  $\mathbb{C}^* \rightarrow G$ . Fix some  $q \in \mathbb{C}^*$  with  $|q| < 1$ . It has been observed by E.Looijenga ([EF, BG]) that there is a bijection between the set of  $q$ -twisted conjugacy classes in the group  $L(G)$  and the set of isomorphism classes of holomorphic  $G$ -bundles on  $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ . This observation suggests to use geometric invariant theory for  $L(G)$  to describe the moduli space of holomorphic principal  $G$  bundles on  $E_q$ . However, in order to obtain a good invariant theory we have to pass to a central extension of the group  $L(G)$ , i.e. to the holomorphic Kac-Moody group  $\tilde{G}$  corresponding to  $G$ . Brüchert [Br] has constructed an analogue of the Steinberg cross section in  $\tilde{G}$ . This cross section carries a natural  $\mathbb{C}^*$ -action so that the quotient is isomorphic to a weighted projective space. Since the Kac-Moody group  $\tilde{G}$  is a central extension of the loop group  $L(G)$ , we can associate to each element of the cross section a holomorphic  $G$  bundle. It turns out that bundles in the same  $\mathbb{C}^*$ -orbit are isomorphic and that the space of  $\mathbb{C}^*$ -orbits is isomorphic to the moduli space of semistable  $G$  bundles over the elliptic curve. The main step here is to show that the bundles coming from the cross section are semistable outside of the section's origin and hence have an image in the moduli space. This is carried out as follows: The origin of the section corresponds to the (twisted) Coxeter element. A careful investigation of its action on the set of all roots of  $G$  shows that the corresponding principal bundle is unstable and has minimal possible automorphism

group dimension among all unstable bundles (Proposition 3.4). Since this bundle is degeneration of all the bundles corresponding to other elements of the cross section, a result of Helmke and Slodowy [HS] implies that the latter bundles have smaller automorphism group dimension and hence are semistable.

The ideas here are mostly due to Peter Slodowy. The Steinberg cross section in  $\tilde{G}$  plays a role in a generalisation of a theorem of Brieskorn which relates simple singularities of type  $A_n$ ,  $D_n$  and  $E_n$  and the corresponding simple Lie groups to the case of elliptic singularities (see [HS1]).

The paper is set up as follows: In section 2 we recall some results from the Theory of affine Kac-Moody Lie algebras and groups. In particular, we describe the Steinberg cross section and mention how some standard results on Kac-Moody groups generalise to the case when the underlying finite-dimensional Lie group is not simply connected. In section 3 we make the connection to holomorphic bundles. Section 4 we give a proof of Proposition 3.4 while the Appendix gives the details of some explicit calculations.

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## 2. HOLOMORPHIC KAC MOODY GROUPS

**2.1. The group.** Let  $G$  be a simple and simply connected algebraic group over  $\mathbb{C}$  and denote by  $L(G)$  group of holomorphic maps from  $\mathbb{C}^* \rightarrow G$  endowed with point-wise multiplication. This is an infinite-dimensional Lie group, called the holomorphic loop group corresponding to  $G$ . The group  $L(G)$  possesses a universal central extension which sits in an exact sequence

$$1 \rightarrow \mathbb{C}^* \xrightarrow{\iota} \hat{L}(G) \xrightarrow{\pi} L(G) \rightarrow 1.$$

Topologically, the group  $\hat{L}(G)$  is a non-trivial  $\mathbb{C}^*$ -bundle over  $L(G)$ . Taking the  $l$ -th power of the corresponding transition functions we obtain the central extension of  $L(G)$  of level  $l$ . Up to isomorphism this yields all central extensions of  $L(G)$  ([PS]).

The natural multiplication action of the multiplicative group  $\mathbb{C}^*$  on the loop group  $L(G)$  lifts to a  $\mathbb{C}^*$ -action on the central extension  $\hat{L}(G)$ , and we define the holomorphic Kac-Moody group corresponding to  $G$  to be the semi-direct product  $\tilde{G} = \hat{L}(G) \rtimes \mathbb{C}^*$ . The Lie algebras of  $G$ ,  $L(G)$  and  $\tilde{G}$  are denoted by  $\mathfrak{g}$ ,  $L(\mathfrak{g})$  and  $\tilde{\mathfrak{g}}$ , respectively.

**2.2. Roots and reflections.** If the finite-dimensional Lie algebra  $\mathfrak{g}$  is simple of rank  $r$ , the subalgebra  $\tilde{\mathfrak{g}}_{\text{pol}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}C \oplus \mathbb{C}D \subset \tilde{\mathfrak{g}}$  of polynomial loops is an untwisted affine Lie algebra in the sense of [K], and  $\tilde{\mathfrak{g}}$  can be viewed as a certain completion of it (see [GW]). Here,  $C$  denotes a generator of the centre of  $\tilde{\mathfrak{g}}_{\text{pol}}$ , and  $D$  is the infinitesimal generator of the  $\mathbb{C}^*$ -action on the centrally extended group  $\hat{L}(G)$ .

Let us fix once and for all a maximal torus  $T \subset G$ , and denote the corresponding Lie algebra  $\text{Lie}(T) = \mathfrak{h} \subset \mathfrak{g}$ . It is known that the Lie algebra  $\tilde{\mathfrak{g}}_{\text{pol}}$  has a root space decomposition with respect to the Cartan subalgebra  $\mathfrak{h} \oplus \mathbb{C}C \oplus \mathbb{C}D \subset \tilde{\mathfrak{g}}_{\text{pol}}$ . Denote the set of roots by  $\tilde{\Delta}$  and fix a set  $\tilde{\Pi} = \{\tilde{\alpha}_0, \dots, \tilde{\alpha}_r\}$  of simple roots. We get a linear combination  $C = \sum_{i=0}^r a_i^\vee \tilde{\alpha}_i^\vee$  where the  $\tilde{\alpha}_i^\vee$  are the simple co-roots. The  $a_i^\vee$  appearing in the expression above are called the dual Kac labels.

Denote the by  $\widetilde{W}$  the Weyl group of the root system  $\widetilde{\Delta}$  and let  $r_i$  be the simple reflection corresponding to  $\widetilde{\alpha}_i \in \widetilde{\Pi}$ . The Weyl group  $\widetilde{W}$  is known to have the following two descriptions:

$$\begin{aligned}\widetilde{W} &\cong W \ltimes Q^\vee, \\ \widetilde{W} &\cong N_{\mathbf{L}(G) \ltimes \mathbb{C}^*}(T \times \mathbb{C}^*) / (T \times \mathbb{C}^*).\end{aligned}$$

Here,  $W$  is the Weyl group of the group  $G$  and  $Q^\vee$  is the co-root lattice of  $\mathfrak{g}$ . Furthermore,  $N_{\mathbf{L}(G) \ltimes \mathbb{C}^*}(T \times \mathbb{C}^*)$  denotes the normaliser of the torus  $T \times \mathbb{C}^*$  in  $\mathbf{L}(G) \ltimes \mathbb{C}^*$ , (see [PS]).

Finally, the product of all simple reflections  $cox = \prod_{i=0}^r r_i$  is called a Coxeter element of  $\widetilde{W}$ . Obviously this definition depends on the choice of a basis of the root system  $\Delta$ . But it is known (see e.g. [Hu]) that all Coxeter elements of  $\widetilde{W}$  are conjugate in  $\widetilde{W}$  unless  $\mathfrak{g}$  is of type  $A_n$ .

**2.3. Representations and characters.** Denote by  $\widetilde{P}$  the weight lattice of  $\widetilde{\mathfrak{g}}_{\text{pol}}$  and by  $\widetilde{P}^+$  its cone of dominant weights which is generated by the fundamental dominant weights  $\lambda_0, \dots, \lambda_r$  and  $\delta$  (throughout this section, we follow the name conventions of [K]). For each  $\lambda \in \widetilde{P}^+$  there is an irreducible highest weight module  $V_\lambda$  of  $\widetilde{\mathfrak{g}}_{\text{pol}}$ . It is known that each of these representations extends to a representation of  $\widetilde{\mathfrak{g}}$  on the analytic completion  $V_\lambda^{an}$  of  $V_\lambda$  which, in turn, lifts to a representation of the holomorphic Kac-Moody group  $\widetilde{G}$  (see [GW]).

The vector space  $V_\lambda$  admits a positive definite Hermitian form  $(\cdot, \cdot)$  which is contravariant with respect to the anti-linear Cartan involution on  $\widetilde{\mathfrak{g}}_{\text{pol}}$ . We denote by  $V_\lambda^{ss}$  the  $L^2$ -completion of  $V_\lambda$  with respect to the norm defined by this Hermitian form.

For a fixed  $q \in \mathbb{C}^*$ , let the “ $q$ -level set” of  $\widetilde{G}$  be the subset  $\widetilde{G}_q = \widehat{\mathbf{L}}(G) \times \{q\} \subset \widetilde{G}$ . Obviously, the  $q$ -level sets  $\widetilde{G}_q$  are invariant under conjugation in  $\widetilde{G}$ .

The following Theorem is known (see [GW], [EFK], [B])

**Theorem 2.1.** *Fix  $q \in \mathbb{C}^*$  such that  $|q| < 1$ . Then we have*

- (i) *For any  $(g, q) \in \widetilde{G}_q$ , the operator  $(g, q) : V_\lambda^{an} \rightarrow V_\lambda^{an}$  uniquely extends to a trace class operator on  $V_\lambda^{ss}$ .*
- (ii) *The function  $\chi_\lambda : \widetilde{G}_q \rightarrow \mathbb{C}$ , defined by  $(g, q) \mapsto \text{tr}|_{V_\lambda^{ss}}(g, q)$  is holomorphic and conjugacy invariant.*
- (iii) *The functions  $\chi_{\lambda_0}, \dots, \chi_{\lambda_r}$  generate the ring of holomorphic conjugacy invariant functions on  $\widetilde{G}_q$ .*

Note that the centre of  $\widetilde{\mathfrak{g}}$  acts on  $V_{\lambda_i}$  by the scalar  $a_i^\vee$ . Hence, for any  $q \in \mathbb{C}^*$  such that  $|q| < 1$ , we can define a conjugacy invariant map  $\chi_q : \widetilde{G}_q \rightarrow \mathbb{C}^{r+1}$  via  $(g, q) \mapsto (\chi_{\lambda_0}(g, q), \dots, \chi_{\lambda_r}(g, q))$ .

**2.4. The cross section.** In this section we shall indicate, following [Br], how to define a section to the map  $\chi_q$ .

Associated to each real root  $\widetilde{\alpha} \in \widetilde{\Delta}$ , (i.e. a root whose  $\widetilde{W}$ -orbit  $\widetilde{W}\widetilde{\alpha}$  contains a simple root,) there exists a one parameter subgroup  $x_{\widetilde{\alpha}} : \mathbb{C} \rightarrow \widehat{\mathbf{L}}(G)$  (see [GW]). For  $0 \leq i \leq r$  and  $c \in \mathbb{C}$ , we set  $x_i(c) = x_{\alpha_i}(c)$ ,  $y_i(c) = x_{-\alpha_i}(c)$ , and  $n_i = x_i(1)y_i(-1)x_i(1)$ . (Thus the element  $n_i \in \widehat{\mathbf{L}}(G)$  is a representative of the simple reflection  $r_i$  in the Weyl group  $\widetilde{W}$ .)

For each  $q \in \mathbb{C}^*$  we can define a map  $\omega_q : \mathbb{C}^{r+1} \rightarrow \tilde{G}_q$  via

$$\omega_q : (c_0, \dots, c_r) \mapsto \left( \prod_{i=0}^r x_i(c_i) n_i, q \right).$$

The map  $\omega_q$  as well as its image  $\mathcal{C}_q = \omega_q(\mathbb{C}^{r+1})$  are called a Steinberg cross-section in  $\tilde{G}_q$ . This name is justified by the analogy of the definition of the map  $\omega_q$  to the usual Steinberg cross section for finite dimensional algebraic groups. The following Theorem is due to Brückert ([Br], Theorem 7):

**Theorem 2.2.** *For  $|q|$  small enough, the map  $\chi_q \circ \omega_q : \mathbb{C}^{r+1} \rightarrow \mathbb{C}^{r+1}$  is an isomorphism of algebraic varieties.*

An essential tool in the proof of Theorem 2.2 is the existence of a  $\mathbb{C}^*$ -action on the cross section  $\mathcal{C}_q$ . Let  $\iota : \mathbb{C}^* \rightarrow \widehat{L}(G)$  denote the identification of the centre of  $\widehat{L}(G)$  with  $\mathbb{C}^*$ . Then we have ([Br], Proposition 15):

**Proposition 2.3.** *There exists a one parameter subgroup  $\mu : \mathbb{C}^* \rightarrow \widehat{L}(G)$  such that for any  $(c, q) \in \mathcal{C}_q$ , we have  $(\mu(z)c\iota(z)\mu(z)^{-1}, q) \in \mathcal{C}_q$ , for all  $z \in \mathbb{C}^*$ .*

Using Proposition 2.3 we can define a  $\mathbb{C}^*$ -action on  $\mathcal{C}_q$  via

$$z : (c, q) \mapsto (\mu(z)c\iota(z)\mu(z)^{-1}, q).$$

The trace function  $\chi_q$  behaves well with respect to this  $\mathbb{C}^*$ -action. In fact, we have ([Br], Theorem 6):

**Proposition 2.4.** *There exists some  $k \in \mathbb{N}$  such that  $\chi_i(z.(c, q)) = z^{ka_i^\vee} \chi_i(c, q)$  for all  $(c, q) \in \mathcal{C}_q$ .*

**Corollary 2.5.** *If we let  $\mathbb{C}^*$  act on  $\mathbb{C}^{r+1}$  with weights  $ka_i^\vee$ , then the restriction of  $\chi_q$  to  $\mathcal{C}_q$  induces a  $\mathbb{C}^*$ -equivariant isomorphism  $\mathcal{C}_q \rightarrow \mathbb{C}^{r+1}$ . In particular, we have*

$$(\mathcal{C}_q - \{\omega(0)\}) / \mathbb{C}^* \cong \mathbb{P}(a_0^\vee, \dots, a_r^\vee),$$

where  $\mathbb{P}(a_0^\vee, \dots, a_r^\vee)$  denotes the weighted projective space with weights  $a_0^\vee, \dots, a_r^\vee$ .

**2.5. The non-simply connected case.** Here we give a brief account on how the constructions of the previous paragraphs generalise to the non simply connected case. A more thorough treatment can be found in [M].

Let  $G$  be a simple algebraic group with universal cover  $\tilde{G}$  with maximal torus  $T$  resp.  $\tilde{T}$  and co-character lattices  $\check{\chi}(T)$  and  $\check{Q} = \check{\chi}(\tilde{T})$ . Denote by  $Z \cong \pi_1(G)$  the kernel of the covering map  $\tilde{G} \rightarrow G$ .

Then the component group of  $L(G)$  is given by  $\pi_0(L(G)) \cong Z$ .

For the representation theory it turns out to be more appropriate to work with the "group of open loops" instead. We set:

$$\mathcal{L}_Z(\tilde{G}) = \{\phi : \mathbb{C} \rightarrow \tilde{G} \text{ such that } \phi(t)\phi(t+1)^{-1} \in Z, \text{ and } \phi \text{ holomorphic}\}.$$

Identifying  $\check{\chi}(T)$  and  $\check{Q}$  as lattices in  $\mathfrak{h}$ , the exponential map provides a group homomorphism:

$$\begin{aligned} \gamma : \check{\chi}(T) &\rightarrow \mathcal{L}_Z(\tilde{G}) \\ \check{\beta} &\mapsto \gamma_{\check{\beta}} : t \mapsto \gamma_{\check{\beta}}(t) := e^{2\pi i \check{\beta} t}. \end{aligned}$$

With these notations the following result holds:

**Lemma 2.6.**

- (i)  $\gamma(\check{Q}) \subset \mathcal{L}_Z(\bar{G})$ ,
- (ii)  $\mathcal{L}_Z(\bar{G}) \cong (\mathcal{L}\bar{G} \rtimes \check{\chi}(T))/\check{Q}$ ,
- (iii)  $\mathcal{L}G \cong \mathcal{L}_Z(\bar{G})/Z$ .

The group  $Z \subset G$  can be canonically identified with a subgroup of the group of diagram automorphisms on the Dynkin diagram of the root system  $\tilde{\Delta}$  (see e.g. [B, T]). Since we are interested in a cyclic component group let us fix an automorphism  $\sigma \in Z$  and denote by  $\Sigma$  the subgroup generated by the element  $\sigma$ . Choose a representative  $\check{\lambda}_\sigma \in \check{\chi}(T)$  of  $\sigma$  and write  $\bar{\Sigma} \cong \mathbb{Z}$  for the subgroup (of  $\check{\chi}(T)$ ) generated by  $\check{\lambda}_\sigma$ . One finds  $(\check{\lambda}_\sigma)^{\text{ord}\sigma} \in \check{Q}$ . Consider the group

$$\mathcal{L}_\Sigma \bar{G} := (\mathcal{L}\bar{G} \rtimes \bar{\Sigma})/\bar{\Sigma}^{\text{ord}\sigma}.$$

In [M], Theorem 3.2, see also [T], Section 3.3, there is a classification of all central extensions of  $\mathcal{L}_\Sigma \bar{G}$ :

**Theorem 2.7.**

- (i) *There is a natural number  $k_f \in \mathbb{N}$  such that for all  $l \in k_f \mathbb{Z}$  a uniquely determined central extension  $\widehat{\mathcal{L}}_\Sigma \bar{G}^l$  of  $\mathcal{L}_\Sigma \bar{G}$  of level  $l$  exists.*
- (ii) *Every central extension of  $\mathcal{L}_\Sigma \bar{G}$  is isomorphic to  $\widehat{\mathcal{L}}_\Sigma \bar{G}^l$  for a certain  $l$ .*
- (iii) *The translation action of  $\mathbb{C}$  on  $\mathcal{L}_\Sigma \bar{G}$  lifts to these central extensions and it factors through  $m\mathbb{Z}$  for a certain  $m \in \mathbb{N}$ .*

Writing  $\widetilde{\mathbb{C}}^*$  for  $\mathbb{C}/m\mathbb{Z}$  this allows us to define the non-connected Kac-Moody group:

**Definition 2.8.** The group  $\tilde{G} := \widehat{\mathcal{L}}_\Sigma \bar{G}^{k_f} \rtimes \widetilde{\mathbb{C}}^*$  is called the affine Kac-Moody group associated to  $G$  and  $\Sigma$ .

If  $\lambda$  is a  $\Sigma$ -invariant dominant weight then its level  $k$  (the multiple by which the centre acts on the representation space) is divisible by  $k_f$ . Furthermore the action of  $\widehat{\mathcal{L}}_\Sigma \bar{G}_0$  extends to a  $\widehat{\mathcal{L}}_\Sigma \bar{G}$ -module with Theorem 2.1 (i) and (ii) still being valid for  $(g, q) \in \widehat{\mathcal{L}}_\Sigma \bar{G}_0$  with  $|q| < 1$ . For  $s+1$  being the number of  $\Sigma$ -orbits on  $\tilde{\Pi}$  there are  $s+1$  dominant weights  $\Lambda_0, \dots, \Lambda_s$  with  $\tilde{P}^{+, \Sigma} := \{\lambda \in \tilde{P}^+, \sigma(\lambda) = \lambda\} = \mathbb{Z}^{\geq 0} < \Lambda_0, \dots, \Lambda_s > \times \mathbb{Z}\delta$ . Write  $\tilde{G}_\sigma$  for the connected component of  $\tilde{G}$  corresponding to  $\sigma$ . Then the following analogue of Theorem 2.1(iii) is valid:

**Proposition 2.9.** *The functions  $\chi_{\Lambda_0}, \dots, \chi_{\Lambda_s}$  generate the ring of holomorphic conjugacy invariant functions on  $\tilde{G}_{\sigma, q}$ , for  $q \in D^*$ .*

Again, this allows us to define the quotient map  $\chi_q : \tilde{G}_{\sigma, q} \rightarrow \mathbb{C}^{s+1}$  via  $(g, q) \mapsto (\chi_{\Lambda_0}(g, q), \dots, \chi_{\Lambda_s}(g, q))$ .

It is also possible to construct a cross section to this quotient map in this situation: Choose representatives  $\{\tilde{\alpha}_0, \dots, \tilde{\alpha}_s\} \subset \tilde{\Pi}$  of the  $\Sigma$ -orbits on  $\tilde{\Pi}$  and a representative  $n_\sigma \in N_G(\hat{T})$  of  $\sigma$ . Here,  $N_G(\hat{T})$  is the normaliser of  $\hat{T}$  in  $\bar{G}$ . Using the notation of section 2.5, define a map:

$$\begin{aligned} \omega_{\sigma, q} : \mathbb{C}^{s+1} &\rightarrow \tilde{G}_{\sigma, q} \\ \omega_{\sigma, q}(c_0, \dots, c_s) &:= \left( \left( \prod_0^s x_i(c_i) n_i \right) n_\sigma, q \right). \end{aligned}$$

We set  $\mathcal{C}_{\sigma,q} := \omega_{\sigma,q}(\mathbb{C}^{s+1})$ . Note that  $\omega_{\sigma,q}(0, \dots, 0)$  is a representative of the twisted Coxeter element  $\text{cox}^\sigma = s_0 \dots s_s \sigma \in \widetilde{W} \rtimes \pi_1(G)$ . (For the definition of twisted Coxeter elements, see [Sp, M].)

It is shown in [M], Theorem 4.2, Lemma 4.1 and Lemma 2.9, that the results Theorem 2.2, Propositions 2.3 and 2.4 and Corollary 2.5 also carry over to this setting, if we replace the dual Kac labels  $a_i^\vee$  by the ones of the orbit Lie algebra in the sense of Fuchs et al. [FSS]. In particular, we have:

**Proposition 2.10.** *For  $|q|$  small enough, the map  $\chi_{\sigma,q} \circ \omega_{\sigma,q} : \mathbb{C}^{s+1} \rightarrow \mathbb{C}^{s+1}$  is an isomorphism of algebraic varieties.*

### 3. HOLOMORPHIC PRINCIPAL G-BUNDLES

**3.1. Gluing maps for G-bundles.** In this section, we relate the conjugacy classes in the  $q$ -level set  $\widetilde{G}_q \subset \widetilde{G}$  to principal  $G$ -bundles over the elliptic curve  $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ . Up to  $C^\infty$ -isomorphism, every principal  $G$ -bundle over  $E_q$  is determined by its topological class, which is an element in  $\pi_1(G) \cong \mathbb{Z}$ . One can classify holomorphic principal  $G$ -bundles of a fixed topological class  $\sigma \in \mathbb{Z}$  as follows. We have  $\pi_0(L(G)) \cong \mathbb{Z}$ . Hence we can consider the connected component  $L(G)_\sigma$  of the loop group  $L(G)$  which corresponds to the element  $\sigma \in \mathbb{Z}$ . The group  $L(G) \rtimes \mathbb{C}^*$  acts on the set  $L(G)_\sigma \times \{q\} \subset L(G) \rtimes \mathbb{C}^*$  by conjugation. A fundamental observation due to E. Looijenga gives a one-to-one correspondence between the set of equivalence classes of holomorphic  $G$ -bundles on  $E_q$  of topological type  $\sigma$  and the set of  $L(G) \times \{1\}$ -orbits in  $L(G)_\sigma \times \{q\}$ . This correspondence comes about as follows. For any element  $(\gamma, q) \in L(G)_\sigma \times \{q\}$  consider the  $G$ -bundle  $\xi_\gamma$  over  $E_q$  which is defined as the quotient  $(\mathbb{C}^* \times G)/\mathbb{Z}$ , where  $\mathbb{Z}$  acts via  $1 : (z, h) \mapsto (qz, \gamma(z)h)$ . Obviously, this construction defines a holomorphic  $G$ -bundle  $\xi_\gamma$  of topological type  $\sigma$  on the elliptic curve  $E_q$ , and the following result due to E. Looijenga is not hard to prove (see e.g. [EF, BG]).

#### Theorem 3.1.

- (i) *Two elements  $(\gamma_1, q), (\gamma_2, q) \in L(G)_\sigma \times \{q\}$  are conjugate under  $L(G) \times \{1\}$  if and only if the corresponding holomorphic  $G$ -bundles  $\xi_{\gamma_1}$  and  $\xi_{\gamma_2}$  are isomorphic.*
- (ii) *For any holomorphic  $G$ -bundle  $\xi \rightarrow E_q$  of topological type  $\sigma$ , there exists an element  $\gamma \in L(G)_\sigma$  such that  $\xi \cong \xi_\gamma$ .*

**3.2. Vector bundles and stability.** Here, we recall the notion of stability of principal bundles over  $E_q$  and the reduction of unstable bundles to a Levi subgroup which corresponds to the Harder-Narasimhan filtration in the vector bundle case. The slope  $\mu(V)$  of a holomorphic vector bundle  $V \rightarrow E_q$  over the elliptic curve  $E_q$  (or over any smooth curve) is defined by  $\mu(V) = \deg(V)/\text{rk}(V)$ . The vector bundle  $V$  is said to be stable (resp. semistable) if

$$\mu(U) < \mu(V) \quad (\text{resp. } \mu(U) \leq \mu(V))$$

for any nontrivial holomorphic subbundle  $U$  of  $V$ .

**Definition 3.2.** A principal bundle  $\xi \rightarrow E_q$  is called semistable if the associated vector bundle  $\text{ad}(\xi) = \xi \times^G \mathfrak{g}$  is a semistable vector bundle. The principal bundle  $\xi \rightarrow E_q$  is called unstable if it is not semistable.

For any holomorphic vector bundle  $V$  on  $E_q$  there exists a unique filtration, the so called Harder Narashiman filtration,  $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$  such that the quotients  $V_i/V_{i-1}$  are semistable and  $\mu(V_i/V_{i-1}) > \mu(V_{i+1}/V_i)$ . Obviously, the bundle  $V$  is semistable exactly if  $n = 1$ . Since  $E_q$  is an elliptic curve, this filtration splits so that we have

$$V \cong \bigoplus V_i/V_{i-1}.$$

The existence of the filtration  $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$  corresponds to a reduction of the structure group of the vector bundle  $V$  from  $GL(V)$  to a parabolic subgroup  $P \subset GL(V)$ , and the splitting of the filtration corresponds to a further reduction of the structure group to a Levi subgroup  $L$  of  $P$ .

This construction generalises to holomorphic  $G$ -bundles as follows (see e.g. [FM1], Theorem 2.7., [HS], Theorem 1.3.1)

**Theorem 3.3.** *Let  $\xi$  be an unstable holomorphic principal  $G$ -bundle over the elliptic curve  $E_q$ . Then there exists a maximal parabolic subgroup  $P$  of  $G$  and a Levi subgroup  $L$  of  $P$  such that the bundle  $\xi$  reduces to a semistable  $L$ -bundle  $\xi_L$  such that for any nonzero dominant character  $\psi : P \rightarrow \mathbb{C}^*$ , the line bundle associated to  $\xi_L$  and  $\psi$  has positive degree.*

**3.3. Bundles for elements in the Steinberg cross section.** Consider the  $q$ -level set  $\tilde{G}_q \subset \tilde{G}$  for some fixed  $q$  with  $|q| < 1$ . The projection  $\pi_q : \tilde{G}_q \rightarrow \mathbf{L}(G) \times \{q\}$ ,  $(g, q) \mapsto (\pi(g), q)$  allows to associate to each element  $(g, q) \in \tilde{G}_q$  the  $G$ -bundle on the elliptic curve  $E_q$  with gluing map  $\pi(g)$ . Fix an element  $\sigma \in \pi_1(G)$ . In this section we determine those  $(g, q) \in \mathcal{C}_{\sigma, q}$  of the Steinberg section whose corresponding principal bundle  $\xi_{\pi(g)}$  (which is of topological type  $\sigma$ ) is semistable.

Assume  $\pi_1(G)$  to be cyclic, possibly trivial, with generator  $\sigma$  and let  $s + 1$  be the number of  $\sigma$ -orbits on the affine Dynkin diagram of  $\tilde{\Delta}$ . Then the following statement holds:

**Proposition 3.4.** *If  $\xi_\sigma$  is the  $G$ -bundle of topological type  $\sigma$  over  $E_q$  corresponding to the twisted Coxeter element then  $\dim \text{Aut}(\xi_\sigma) = s + 1$ .*

We give a proof of this Proposition in section 4.

*Remark 3.5.* Using the terminology of [FM2] we have shown that the twisted Coxeter element gives rise to a minimally unstable bundle.

As before, we denote the Steinberg cross section by  $\mathcal{C}_{\sigma, q} = \omega_{\sigma, q}(\mathbb{C}^{s+1})$  and set  $\mathcal{C}_{\sigma, q}^* = \mathcal{C}_{\sigma, q} \setminus \{\omega_{\sigma, q}(0)\}$ . Note that Proposition 3.4 implies the following corollary:

**Corollary 3.6.**

- (i) *The bundle  $\xi_\sigma$  is unstable with minimal possible automorphism group dimension.*
- (ii) *All bundles corresponding to elements in  $\mathcal{C}_{\sigma, q}^*$  are semistable.*

*Proof.* Using the construction from section 3.1, we associate to each element of the Steinberg cross section  $\mathcal{C}_{\sigma, q}$  a holomorphic  $G$ -bundle on  $E_q$  of topological type  $\sigma$ .

Consider the following family  $\tilde{\Xi}$  of  $G$ -bundles over the pointed complex space  $(\mathcal{C}_{\sigma, q}, \omega_{\sigma, q}(0))$ :

$$\tilde{\Xi} = (\mathbb{C}^* \times \mathcal{C}_{\sigma, q} \times G)/\mathbb{Z},$$

with  $1 \in \mathbb{Z}$  acting by:

$$1 : (z, (g, q), h) \mapsto (qz, (g, q), \pi(g)(z)h).$$

As usual,  $\pi$  denotes the projection of the centrally extended loop group to the loop group itself. The special fibre of this family is the bundle  $\xi_\sigma$  from Proposition 3.4 and the construction shows that this bundle is adjacent to all the other bundles  $\eta$  arising from elements in the Steinberg section  $\mathcal{C}_{\sigma,q}^*$ . (A bundle  $\xi$  is said to be adjacent to a bundle  $\eta$  if there is a family of  $G$ -bundles over a pointed complex space  $(\mathcal{B}, b)$  with  $\xi$  being isomorphic to the special fibre and each other fibre being isomorphic to  $\eta$ .) Here we use that the section admits a  $\mathbb{C}^*$ -action with  $\omega_{\sigma,q}(0)$  being contained in every orbit closure. By [HS] Proposition 3.9 this implies  $\dim \text{Aut}(\eta) \leq s$ . In view of [HS], Proposition 3.3 and Theorem 2.2, the each bundle  $\eta \in \mathcal{C}_{\sigma,q}^*$  has to be (regular) semistable.  $\square$

*Remark 3.7.* Using Atiyah's classification of vector bundles on an elliptic curve, one can show that if the group  $G$  is simply connected then the  $G$ -bundle  $\xi_g$  corresponding to an element  $(g, q) \in \tilde{G}_q$  is unstable exactly if  $\chi_q(g, q) = 0$ . In particular, the bundles corresponding to Weyl group elements of infinite order are unstable. Presumably this statement also holds for the non-simply connected case.

**3.4. The moduli space of holomorphic  $G$ -bundles.** In this section we aim to describe the coarse moduli space  $\mathcal{M}_{\sigma,q}(G)$  of semistable  $G$ -bundles over the elliptic curve  $E_q$  of topological type  $\sigma \in \pi_1(G)$  where we assume that  $\sigma$  generates  $\pi_1(G)$ . We keep the notation of the previous section. If  $(g_1, q)$  and  $(g_2, q)$  are two elements of  $\mathcal{C}_{\sigma,q}^*$  which lie in the same  $\mathbb{C}^*$ -orbit on  $\mathcal{C}_{\sigma,q}^*$ , then the bundles corresponding to  $(g_1, q)$  and  $(g_2, q)$  are isomorphic. In the non-simply connected case keep in mind that the loop group  $\mathcal{L}G$  is obtained from the group of open loops  $\mathcal{L}_Z(\tilde{G})$  by further division by  $Z = \pi_1(G)$ . However, a computation using the isomorphism given by Proposition 2.10 shows that the product  $(cg, q)$  of an element  $c \in Z$  and an element  $(g, q) \in \mathcal{C}_{\sigma,q}^*$  is conjugate to an element in the  $\mathbb{C}^*$ -orbit of  $\xi$ .

Therefore we obtain a map

$$\tilde{\psi} : \mathcal{C}_{\sigma,q}^* \rightarrow \mathcal{M}_{\sigma,q}(G),$$

which factors through an injective map

$$\psi : \mathcal{C}_{\sigma,q}^*/\mathbb{C}^* \rightarrow \mathcal{M}_{\sigma,q}(G).$$

It remains to show that  $\psi$  is an algebraic isomorphism. To this end, let us construct a holomorphic  $G$ -bundle  $\Xi$  on  $E_q \times \mathcal{C}_{\sigma,q}^*$  as the quotient

$$\Xi = (\mathbb{C}^* \times \mathcal{C}_{\sigma,q}^* \times G)/\mathbb{Z},$$

where  $\mathbb{Z}$  acts via

$$1 : (z, (g, q), h) \mapsto (qz, (g, q), \pi(g)(z)h).$$

This shows that the map  $\mathcal{C}_{\sigma,q}^* \rightarrow \mathcal{M}_{\sigma,q}(G)$  is algebraic. Consequently, the induced map  $\psi : \mathcal{C}_{\sigma,q}^*/\mathbb{C}^* \rightarrow \mathcal{M}_{\sigma,q}(G)$  is algebraic as well.

Now,  $\psi$  is an injective morphism of irreducible projective varieties of the same dimension. So it has to be an isomorphism. Hence we have proved

**Theorem 3.8.** *Let  $\mathcal{C}_{\sigma,q}^* = \omega_{\sigma,q}(\mathbb{C}^{s+1} - \{0\})$  denote the cross section in  $\tilde{G}_{\sigma,q}$  and let  $\mathcal{M}_{\sigma,q}(G)$  denote the moduli space of semistable  $G$ -bundles on the elliptic curve  $E_q$ . Then*

$$\mathcal{M}_{\sigma,q}(G) \cong \mathcal{C}_{\sigma,q}^*/\mathbb{C}^*.$$

In particular, using Corollary 2.5, respectively its analogue in the non-simply connected case (see section 2.5) we get the following well known result (see e.g. [FM2], [L] and [S] for the non-simply connected case).

**Corollary 3.9.** *Let  $a_0^\vee, \dots, a_s^\vee$  denote the dual Kac-labels of  $G$  (respectively, the dual Kac-labels of the orbit Lie algebra of  $G$ ) for  $\sigma \neq \text{id}$ . Then*

$$\mathcal{M}_{\sigma,q}(G) \cong \mathbb{P}(a_0^\vee, \dots, a_s^\vee).$$

#### 4. PROOF OF PROPOSITION 3.4

*Proof.* We have to show that the automorphism group  $\text{Aut}(\xi_\sigma)$  of the bundle corresponding to the twisted Coxeter element has dimension  $s+1$ . This is carried out by a case-by-case analysis. The corresponding calculations will be carried out over the adjoint bundle  $\text{ad}(\xi_\sigma) = \xi_\sigma \times^G \mathfrak{g}$ . It is easy to see that  $\text{ad}(\xi_\sigma)$  can be described as the quotient  $(\mathbb{C}^* \times \mathfrak{g})/\mathbb{Z}$  where  $1 \in \mathbb{Z}$  acts by  $(z, X) \mapsto (zq, \text{Ad}(\gamma_\sigma(z))(X))$ . Here  $\gamma_\sigma$  is the open loop corresponding to the twisted Coxeter element.

**Step 1:** We first determine the Harder-Narasimhan filtration of the bundle  $\xi_\sigma$ . This is achieved by considering the action of  $\gamma_\sigma$  on the Lie algebra. One easily sees that subbundles of  $\text{ad}(\xi_\sigma)$  correspond to subspaces of  $\mathfrak{g}$  which are invariant under  $\text{Ad}(\gamma_\sigma(z))$  for all  $z \in \mathbb{C}^*$ . After having fixed a basis for the root system, the loop  $\gamma_\sigma$  has the form:  $\gamma_\sigma = n_0 \dots n_s \check{\lambda}(z) n_w$  for our representatives of the simple reflections  $s_i$  (with exactly one reflection chosen for each  $\sigma$ -orbit) and an element  $w \in W$ , and a simple co-weight  $\check{\lambda}$ . The elements  $\check{\lambda}$  in the dual weight lattice and  $w \in W$  represent the automorphism  $\sigma$  in the following sense: Consider the extended affine Weyl group  $\widehat{W} = \Lambda^\vee \rtimes W$  with  $\Lambda^\vee$  being the co-root lattice of the finite dimensional Lie algebra  $\mathfrak{g}$ . Then, following [B, T] the stabiliser  $\widehat{W}_A$  of the fundamental alcove  $A$ , for its definition see Kac [K], is naturally isomorphic to  $\Lambda^\vee/Q^\vee$  with  $Q^\vee$  being the co-root lattice. Under this identification we have  $\sigma = \check{\lambda}w$ . For an explicit description of  $\check{\lambda}$  and  $w$ , see [T], Proposition 4.1.4. If  $\sigma = \text{id}$  we have:  $w = \text{id}$  and  $\check{\lambda} = s_r \dots s_1 s_\theta(\check{\theta})$ , where  $\theta$  is the highest root of the finite dimensional Lie algebra  $\mathfrak{g}$ . This leads us to the following decomposition of  $\mathfrak{g}$ :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mathcal{O} \subset R} \mathfrak{g}_{\mathcal{O}}$$

where  $\mathcal{O}$  runs through all  $s_0 \dots s_s w$ -orbits of  $R$  and  $\mathfrak{g}_{\mathcal{O}}$  is defined by:

$$\mathfrak{g}_{\mathcal{O}} = \bigoplus_{\alpha \in \mathcal{O}} \mathfrak{g}_\alpha$$

Since  $\check{\lambda}(z)$  acts trivially on  $\mathfrak{h}$  the corresponding  $\gamma_\sigma$ -invariant subspaces of  $\mathfrak{h}$  are direct sums of eigen-spaces of  $s_0 \dots s_s w$ . These are easily seen to provide subbundles of degree zero. For a given orbit  $\mathcal{O}$  the loop  $\check{\lambda}(z)$  acts diagonally on the root spaces while  $s_0 \dots s_s w$  permutes them cyclically. Therefore, the degree of the subbundle corresponding to  $\mathfrak{g}_{\mathcal{O}}$  is given by  $d_{\mathcal{O}} = \sum_{\alpha \in \mathcal{O}} \alpha(w^{-1}(\check{\lambda}))$ . Denote by  $\Gamma$  the group generated by  $s_0 \dots s_s w$  and by  $\Gamma_\alpha$  the stabiliser of a given root  $\alpha$ . Note that the action of  $s_0 \dots s_s w$  coincides with the action of  $\text{cox}^\sigma$  on  $\mathfrak{h}$ . Writing  $\check{b} = \sum_{i=1}^{|\Gamma|} (\text{cox}^\sigma)^i(w^{-1}(\check{\lambda}))$  we obtain  $d_{\mathcal{O}} = \frac{1}{|\Gamma_\alpha|} \alpha(\check{b})$  for any  $\alpha \in \mathcal{O}$ . The co-weight  $\check{b}$  is  $\text{cox}^\sigma$ -invariant and non-zero. Indeed, by [M], Corollary 2.6 and Proposition 2.7, we know that  $\text{cox}^\sigma$  is of infinite order on  $\mathfrak{h} \oplus \mathbb{C}C$  the Cartan subalgebra of the centrally extended loop group. Since  $s_0 \dots s_s w$  has finite order the element  $(\text{cox}^\sigma)^{|\Gamma|}$  of the extended affine

Weyl group has to be a non-vanishing translation which is easily calculated to be  $\check{b}$ . The invariance property is clear by definition. Applying [M] Proposition 2.8 (and calculating modCC)  $\check{b}$  has to coincide, up to a non-vanishing factor, with the solution  $b$  to the equation  $(\text{cox}^\sigma - 1)(b) = kC$ . Thus we obtain a partition of the root system:

$$\begin{aligned} (1) \quad R_{\check{b}}^+ &= \{\alpha \in R \mid \alpha(\check{b}) > 0\} \\ (2) \quad R_{\check{b}}^- &= \{\alpha \in R \mid \alpha(\check{b}) < 0\} \\ (3) \quad R_{\check{b}}^0 &= \{\alpha \in R \mid \alpha(\check{b}) = 0\} \end{aligned}$$

Therefore the root sub-system  $R_{\check{b}}^0$  describes the root system of some reduction Levi subgroup  $L(\check{b}) = C_G(\mathbb{C}\check{b})$  and  $R_{\check{b}}^0 \cup R_{\check{b}}^+$  that of the parabolic subgroup  $P(\check{b})$ . If this is not the Harder-Narasimhan reduction, then the Levi subgroup is too big and hence there has to be some orbit  $\mathcal{O} \in R_{\check{b}}^0$  and a subspace  $\mathfrak{a} \subset \mathfrak{g}_{\mathcal{O}}$  such that the bundle  $\mathfrak{A} \subset \text{ad}\xi_\sigma$  corresponding to  $\mathfrak{a}$  has positive degree. This implies that  $0 < \dim H^0(E_q, \mathfrak{A}) \leq \dim H^0(E_q, \mathfrak{g}_{\mathcal{O}})$  where  $\mathfrak{g}_{\mathcal{O}}$  is the bundle corresponding to  $\mathfrak{g}_{\mathcal{O}}$ . However, in step 2 below we will prove  $\dim H^0(E_q, \mathfrak{g}_{\mathcal{O}}) = 0$ .

A case-by-case investigation whose basic results are summarised in the appendix shows:

**Lemma 4.1.** *The parabolic subgroup  $P(\check{b})$  is maximal and  $s_0 \dots s_s w$  is the Coxeter element of its Levi  $L(\check{b})$ . This Levi coincides with the Levi corresponding to an unstable bundle of least possible automorphism group dimension (see [HS], Section 6.)*

**Step 2:** Next, we have to calculate the dimension of the automorphism group  $\text{Aut}_G(\xi_\sigma)$ . According to [HS] Proposition 2.4 this group has a semidirect product structure:  $\text{Aut}_G(\xi_\sigma) = \text{Aut}_{L(\check{b})}(\xi_{\sigma L(\check{b})}) \ltimes \text{Aut}_G(\xi_\sigma)^+$ . The group  $\text{Aut}_G(\xi_\sigma)^+$  is the connected unipotent group with Lie algebra  $\text{Lie Aut}_G(\xi_\sigma)^+ = H^0(E_q, \xi_{\sigma L(\check{b})} \times^{L(\check{b})} \mathfrak{n})$  where  $\xi_{\sigma L(\check{b})} \times^{L(\check{b})} \mathfrak{n}$  denotes the associated vector bundle bundle with fibre  $\mathfrak{n}$  being the Lie algebra of the nilpotent radical of the Harder-Narasimhan parabolic subgroup. The formulae in [HS] preceding Proposition 2.4 of [HS] imply

$$\dim \text{Aut}_G(\xi_\sigma)^+ = \sum_{\mathcal{O} \subset R^+ \setminus R(L(\check{b}))} d_{\mathcal{O}}.$$

These dimensions  $\dim \text{Aut}_G(\xi_\sigma)^+$  turn out to be equal to  $s$ . For the readers convenience they are compiled in the appendix.

This leaves us with showing  $\dim \text{Aut}_{L(\check{b})}(\xi_{\sigma L(\check{b})}) = 1$ . Using the equality  $\text{Lie Aut}_{L(\check{b})}(\xi_{\sigma L(\check{b})}) = H^0(E_q, \text{ad}(\xi_{\sigma L(\check{b})}))$  and lifting the sections to the trivial Lie  $L(\check{b})$ -bundles over  $\mathbb{C}^*$  we have to determine the solutions  $h \in \text{L}(\text{Lie } L(\check{b}))$  of the equation

$$\text{Ad}(\gamma_\sigma(z))(h(z)) = h(qz).$$

Without loss of generality we can assume that  $h$  takes values either in an eigen-space of the Coxeter element  $\text{cox}_L$  or in a space  $\mathfrak{g}_{\mathcal{O}}$  with  $\mathcal{O} \subset R(L(\check{b}))$ . In the first case we are left with the equation  $\zeta h(z) = h(qz)$  where  $\zeta$  is the corresponding eigenvalue. The only non-vanishing solution of this equation occur for  $\zeta = 1$  and constant functions on  $E_q$ . The corresponding eigen-space coincides with the one-dimensional centre of  $\text{Lie } L(\check{b})$ . Labelling the elements of the orbit  $\mathcal{O} = \{\beta_1, \dots, \beta_p\}$

for  $p = |\mathcal{O}|$  and setting  $h(z) = \sum_{i=1}^p h_i(z) \otimes x_{\beta_i}$  where  $x_{\beta_i}$  is a basis element of the corresponding root space we require:

$$z^{(\beta_i, w^{-1}(\check{\lambda}))} h_i(z) = h_{i+1}(qz).$$

Iterating this formula we obtain:

$$h_1(q^p z) = q^{\sum_{i=1}^p (p-i)(\beta_i, w^{-1}(\check{\lambda}))} z^{d_{\mathcal{O}}} h_1(z).$$

A tedious calculation using  $d_{\mathcal{O}} = 0$  shows that the exponent of  $q$  on the right hand side is not divisible by  $p$ . Therefore, this functional equation does not permit a solution on  $\mathbb{C}^*$ .  $\square$

*Remark 4.2.* (i) This case-by-case line of reasoning works for simply connected structure groups and similar phenomena occur (the role of  $\check{b}$ , the restriction of the Coxeter element of the affine group yielding the Coxeter element of the Levi from the reduction, etc.). We refrain from giving the detail in favour of the more uniform and elegant treatment we use in this case.  
(ii) It would be interesting to find a more conceptual reason of the fact that the restriction of the twisted Coxeter element to  $\mathfrak{h}$  gives the Coxeter element of the Levi  $L(\check{b})$ .

## 5. APPENDIX

Here we summarise the explicit results of the case-by-case calculations. Let us label the vertexes of the Dynkin diagram as in [B]. For the diagram automorphisms we use the following conventions. In all cases except  $D_{2n}$ ,  $\gamma$  is the generator of the  $\pi_1(G^{ad})$ , where  $G^{ad}$  is the adjoint group. In the  $D_{2n}$ -case,  $\gamma^2$  generates the fundamental group of  $SO_{4n}$  and  $\tau \neq id$  generates  $\pi_1(G)$  with  $\tau^2 = id$  and  $G$  is not isomorphic to  $SO_{4n}$ . We only indicate  $d_{\mathcal{O}}$  for orbits containing the basis elements because  $d_{\mathcal{O}}$  for all other orbits can be derived from these. (Note, that taking the sum over elements of two orbits might change the orbit length!) In the sequel  $\alpha_0$  denotes the negative highest root of the root system  $\Delta$  of  $G$  corresponding to the basis  $\Pi = \{\alpha_1, \dots, \alpha_r\}$ .

**A<sub>n</sub>** : For  $\sigma = id$  the cox-orbits involving the basis elements look as follows:

$\mathcal{O}_1 = \{\alpha_1, \dots, \alpha_{n-1}, -(\alpha_1 + \dots + \alpha_{n-1})\}$  and  $\mathcal{O}_2 = \{\alpha_n, \alpha_{n-1} + \alpha_n, \dots, \alpha_1 + \dots + \alpha_n\}$ . Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = 0$  and  $d_{\mathcal{O}_2} = n+1$ . Hence,  $P(\check{b}) = P_{\alpha_n}$  and  $\dim \text{Aut}_G(\xi_{\sigma})^+ = n+1$

For  $\sigma = \gamma$  we have  $\check{\lambda} = \check{\lambda}_1$  and  $w = s_1 \dots s_n$ . Thus,  $\text{cox}^{\sigma} = s_1 \check{\lambda}_1 s_1 \dots s_n$ . Considering the cox $^{\sigma}$ -orbits involving the basis elements yields:

$\mathcal{O}_1 = \{\alpha_2, \dots, \alpha_n, -(\alpha_2 + \dots + \alpha_n)\}$  and  $\mathcal{O}_2 = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_n\}$ . Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = 0$  and  $d_{\mathcal{O}_2} = -1$ . Hence,  $P(\check{b}) = P_{-\alpha_1}$  and  $\dim \text{Aut}_G(\xi_{\sigma})^+ = 1$

For  $\sigma = \gamma^l$  with  $l|n$  we have  $\text{cox}^{\sigma} = s_1 \dots s_l \check{\lambda}_l w^l$ . We obtain the same results as above except for  $d_{\mathcal{O}_2}$  which yields  $d_{\mathcal{O}_2} = -l$  and  $\dim \text{Aut}_G(\xi_{\sigma})^+ = l$ .

**B<sub>n</sub>** : Consider  $\sigma = id$ . For displaying the correct parabolic subgroup in standard form it turns out that a change of the basis of the root system is necessary:

$\beta_i = \alpha_{n-i}$ ,  $1 \leq i \leq n-1$  and  $\beta_n = -(\alpha_1 + \dots + \alpha_n)$ .

Then, the cox-orbits involving the basis elements have the following form:

$\mathcal{O}_1 = \{\beta_1, \dots, \beta_{n-2}, -(\beta_1 + \dots + \beta_{n-2})\}$ ,  $\mathcal{O}_2 = \{\beta_n, -\beta_n\}$  and for  $n$  odd  $\mathcal{O}_3 = \{\beta_{2i} + \dots + \beta_{n-1}, 1 \leq i \leq \frac{n-1}{2}\} \cup \{\beta_{2i-1} + \dots + \beta_{n-1} + 2\beta_n, 1 \leq i \leq \frac{n-1}{2}\}$ , respectively

for  $n$  even  $\mathcal{O}_3 = \{\beta_i + \dots + \beta_{n-1}, 1 \leq i \leq n-1\} \cup \{\beta_i + \dots + \beta_{n-1} + 2\beta_n, 1 \leq i \leq n-1\}$ . Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = 0$ ,  $d_{\mathcal{O}_2} = 0$  and  $d_{\mathcal{O}_3} = -1$ , for odd  $n$  respectively  $d_{\mathcal{O}_3} = -2$  in the even case. Hence,  $P(\check{b}) = P_{-\beta_{n-1}}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = n+1$ .

Consider  $\sigma = \gamma$ . We have  $\check{\lambda} = \check{\lambda}_1$  while  $w$  fixes  $\alpha_2, \dots, \alpha_n$  and interchanges  $\alpha_1$  with  $\alpha_0$ . Hence,  $\text{cox}^\sigma = s_n \dots s_1 \check{\lambda}_1 w$ . The  $\text{cox}^\sigma$ -orbits involving the basis elements have the following shape:

$\mathcal{O}_1 = \{\alpha_1, \dots, \alpha_{n-1}, -(\alpha_1 + \dots + \alpha_{n-1})\}$  and  $\mathcal{O}_2 = \{\alpha_n, \alpha_{n-1} + \alpha_n, \dots, \alpha_1 + \dots + \alpha_n\}$ . Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = 0$  and  $d_{\mathcal{O}_2} = -1$ . Hence,  $P(\check{b}) = P_{-\alpha_n}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = n$ .

**C<sub>n</sub>** : For  $\sigma = id$  the cox-orbits involving the basis elements look as follows:  
 $\mathcal{O}_1 = \{\alpha_1, \dots, \alpha_{n-1}, -(\alpha_1 + \dots + \alpha_{n-1})\}$ ,  $\mathcal{O}_2 = \{2(\alpha_1 + \dots + \alpha_{n-1}) + \alpha_n, \dots, 2\alpha_{n-1} + \alpha_n, \alpha_n\}$ .

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = 0$ ,  $d_{\mathcal{O}_2} = 2$ , Hence,  $P(\check{b}) = P_{\alpha_n}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = n+1$ .

For  $\sigma = \gamma$  we have  $\check{\lambda} = \check{\lambda}_n$  and  $w$  interchanging  $\alpha_i$  with  $\alpha_{n-i}$  with  $\alpha_0$  being the negative of the highest root of the finite dimensional Lie algebra  $\mathfrak{g}$ . Thus,  $\text{cox}^\sigma = s_{[\frac{n+1}{2}]} \dots s_n \check{\lambda}_n w$ . For displaying the correct parabolic subgroup in standard form it turns out that a change of the basis of the root system is necessary:

$$\beta_i = \alpha_{n-i}, 1 \leq i \leq [\frac{n-1}{2}] - 1, \beta_{[\frac{n-1}{2}]} = -(\alpha_1 + \dots + \alpha_{[\frac{n}{2}]} + 2(\alpha_{[\frac{n}{2}]+1} + \dots + \alpha_{n-1}) + \alpha_n)$$

$$\beta_i = \alpha_{i-[\frac{n-1}{2}]} \text{ for } [\frac{n+1}{2}] \leq i \leq n-2, \beta_n = \alpha_{[\frac{n}{2}]} \text{ and } \beta_n = 2(\alpha_{[\frac{n}{2}]+1} + \dots + \alpha_{n-1}) + \alpha_n$$

For even  $n$  the  $\text{cox}^\sigma$ -orbits involving the basis elements are given by:  
 $\mathcal{O}_1 = \{\beta_1, \dots, \beta_{n-2}, -(\beta_1 + \dots + \beta_{n-2})\}$ ,  $\mathcal{O}_2 = \{\beta_n, -\beta_n\}$  and  $\mathcal{O}_3 = \{\beta_i + \dots + \beta_{n-1}, 1 \leq i \leq n-1\} \cup \{\beta_i + \dots + \beta_n, 1 \leq i \leq n-1\}$ .

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = 0$ ,  $d_{\mathcal{O}_2} = 0$  and  $d_{\mathcal{O}_3} = -1$ . Hence,  $P(\check{b}) = P_{-\beta_{n-1}}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = [\frac{n}{2}] - 1$ .

In the odd case we calculate:

$\mathcal{O}_1 = \{\beta_1, \dots, \beta_{n-1}, -(\beta_1 + \dots + \beta_{n-1})\}$  and  $\mathcal{O}_2 = \{2(\beta_i, \dots, \beta_{n-1}) + \beta_n, 1 \leq i \leq n\}$ . The degrees are given by  $d_{\mathcal{O}_1} = 0$  and  $d_{\mathcal{O}_2} = -1$  yielding  $P(\check{b}) = P_{-\beta_n}$ .

**D<sub>n</sub>** : Consider  $\sigma = id$ . Also, here we have to introduce a new basis of the root system:

$$\beta_i = \alpha_{n-i-1}, 1 \leq i \leq n-2, \beta_{n-1} = -(\alpha_1 + \dots + \alpha_{n-1}) \text{ and } \beta_n = -(\alpha_1 + \dots + \alpha_{n-2} + \alpha_n).$$

Then, the cox-orbits involving the basis elements have the following form:

$\mathcal{O}_1 = \{\beta_1, \dots, \beta_{n-3}, -(\beta_1 + \dots + \beta_{n-3})\}$ ,  $\mathcal{O}_2 = \{\beta_{n-1}, -\beta_{n-1}\}$ ,  $\mathcal{O}_3 = \{\beta_n, -\beta_n\}$  and for even  $n$   $\mathcal{O}_4 = \{\beta_{2i} + \dots + \beta_{n-2}, 1 \leq i \leq \frac{n-2}{2}\} \cup \{\beta_{2i-1} + \dots + \beta_{n-1} + \beta_n, 1 \leq i \leq \frac{n-2}{2}\}$ , respectively for  $n$  odd  $\mathcal{O}_4 = \{\beta_i + \dots + \beta_{n-2}, 1 \leq i \leq n-1\} \cup \{\beta_i + \dots + \beta_{n-1} + \beta_n, 1 \leq i \leq n-1\}$ .

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = d_{\mathcal{O}_2} = d_{\mathcal{O}_3} = 0$  and  $d_{\mathcal{O}_4} = -1$ , for even  $n$  respectively  $d_{\mathcal{O}_4} = -2$  in the odd case. Hence,  $P(\check{b}) = P_{-\beta_{n-2}}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = n+1$ .

For  $\sigma = \gamma^2$  i.e.  $G = SO_{2n}$  we have  $\check{\lambda} = \check{\lambda}_1$  and  $w$  simultaneously interchanges  $\alpha_0$  with  $\alpha_1$  and  $\alpha_{n-1}$  with  $\alpha_n$ . Thus,  $\text{cox}^\sigma = s_n \dots s_1 \check{\lambda}_1 w$ . The  $\text{cox}^\sigma$ -orbits involving the basis elements have the following shape:

$\mathcal{O}_1 = \{\alpha_1, \dots, \alpha_{n-2}, \alpha_n, -(\alpha_1 + \dots + \alpha_{n-2} + \alpha_n)\}$  and  $\mathcal{O}_2 = \{\alpha_i + 2(\alpha_{i+1} + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n, 1 \leq i \leq n-2\} \cup \{\alpha_{n-1}, \alpha_1 + \dots + \alpha_{n-1}\}$ .

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = 0$  and  $d_{\mathcal{O}_2} = -2$ . Hence,  $P(\check{b}) = P_{-\alpha_{n-1}}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = n-1$ .

Consider  $n$  odd and  $\sigma = \gamma$ . Then  $\check{\lambda} = \check{\lambda}_n$  and  $w$  permutes  $\alpha_0, \alpha_n, \alpha_1$  and  $\alpha_{n-1}$  cyclically while interchanging  $\alpha_i$  with  $\alpha_{n-i}$  for the other labels. We have  $\text{cox}^\sigma = s_{\frac{n+1}{2}} \dots s_n \check{\lambda}_n w$ . Again we have to find a new basis in order to get a parabolic in standard form:

$$\beta_i = \alpha_{i+1} \text{ for } 1 \leq i \leq \frac{n-1}{2} - 1, \beta_{\frac{n-1}{2}} = \alpha_{\frac{n+1}{2}} + \dots + \alpha_n, \beta_i = \alpha_{\frac{3n-1}{2}-i}, \frac{n+1}{2} \leq i \leq n-2, \\ \beta_{n-1} = \alpha_1 + \dots + \alpha_{\frac{n+1}{2}} \text{ and } \beta_n = -(\alpha_1 + \dots + \alpha_{\frac{n+1}{2}} + 2(\alpha_{\frac{n+1}{2}+1} + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n).$$

The orbits containing simple roots look like:

$$\mathcal{O}_1 = \{\beta_1, \dots, \beta_{n-1}, -(\beta_1 + \dots + \beta_{n-1})\} \quad \mathcal{O}_2 = \{\beta_i + 2(\beta_{i+1} + \dots + \beta_{n-2}) + \beta_{n-1} + \beta_n, 1 \leq i \leq n-2\} \cup \{\beta_n, \beta_1 + \dots + \beta_{n-2} + \beta_n\}$$

For the degrees we calculate  $d_{\mathcal{O}_1} = 0$  and  $d_{\mathcal{O}_2} = 2$  yielding  $P(\check{b}) = P_{-\beta_n}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = \frac{n-1}{2}$ .

Let us turn to even  $n$  and  $\sigma = \tau$ . Then,  $\check{\lambda} = \check{\lambda}_n$  and  $w$  interchanges  $\alpha_i$  and  $\alpha_{n-i}$  while  $\text{cox}^\sigma = s_{\frac{n}{2}} \dots s_n \check{\lambda}_n w$ . Again we have to find a new basis in order to get a parabolic in standard form:

$$\beta_i = \alpha_{i+1+\frac{n}{2}} \text{ for } 1 \leq i \leq \frac{n}{2} - 3, \beta_{\frac{n}{2}-2} = \alpha_{\frac{n}{2}} + \dots + \alpha_n, \beta_i = \alpha_{n-i-2} \text{ for } \frac{n}{2} - 1 \leq i \leq n-4, \beta_{n-3} = \alpha_1, \beta_{n-2} = -(\alpha_1 + \dots + \alpha_{\frac{n}{2}}), \beta_{n-1} = -(\alpha_{\frac{n}{2}+1} + \dots + \alpha_{n-1}) \text{ and} \\ \beta_n = -(\alpha_{\frac{n}{2}+1} + \dots + \alpha_{n-2} + \alpha_n).$$

We obtain the following orbits containing simple roots:

$$\mathcal{O}_1 = \{\beta_1, \dots, \beta_{n-4}, -(\beta_1 + \dots + \beta_{n-4})\} \quad \mathcal{O}_2 = \{\beta_{n-2}, \beta_{n-1}, \beta_n, -(\beta_{n-2} + \beta_{n-1} + \beta_n)\} \\ \text{and } \mathcal{O}_3 = \{\beta_i + \dots + \beta_{n-3} + r_j, 1 \leq i \leq n-3, r_j \in \{0, \beta_{n-2} + \beta_{n-1}, \beta_{n-2} + \beta_n, \beta_{n-2} + \beta_{n-1} + \beta_n, \dots\}\}.$$

The degrees are given by  $d_{\mathcal{O}_1} = d_{\mathcal{O}_2} = 0$  and  $d_{\mathcal{O}_3} = 2$  yielding  $P(\check{b}) = P_{\beta_{n-3}}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = \frac{n}{2} + 1$ .

**E<sub>6</sub>:** For  $\sigma = id$  we have to find a new basis of the root system:

$$\beta_1 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6), \beta_2 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \beta_3 = \alpha_1, \\ \beta_4 = \alpha_2 + \alpha_3 + \alpha_6, \beta_5 = \alpha_3 + \alpha_4 \text{ and } \beta_6 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6).$$

Then, the cox-orbits involving the basis elements have the following form:

$$\mathcal{O}_1 = \{\beta_1, \beta_2, -\beta_1 - \beta_2\}, \mathcal{O}_2 = \{\beta_4, \beta_5, -\beta_4 - \beta_5\}, \mathcal{O}_3 = \{\beta_6, -\beta_6\} \text{ and } \mathcal{O}_4 = \\ \{\beta_3, \beta_3 + \beta_6, \beta_2 + \beta_3 + \beta_4, \beta_2 + \beta_3 + \beta_4 + \beta_6, \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5, \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6\}.$$

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = d_{\mathcal{O}_2} = d_{\mathcal{O}_3} = 0$  and  $d_{\mathcal{O}_4} = -1$ . Hence,  $P(\check{b}) = P_{-\beta_3}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = 7$ .

Consider  $\sigma = \gamma$ . Then,  $\check{\lambda} = \check{\lambda}_6$  and  $w$  permutes  $\alpha_0, \alpha_1, \alpha_6$  respectively  $\alpha_2, \alpha_3, \alpha_5$  cyclically while fixing  $\alpha_4$ . The twisted Coxeter element has the form  $s_1 s_3 s_4 \check{\lambda}_6 w$ .

Also here we need new basis of the root system:

$$\beta_1 = \alpha_5 + \alpha_6, \beta_2 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \beta_3 = \alpha_3 + \alpha_4, \beta_4 = \alpha_2, \beta_5 = \alpha_0 = \\ -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) \text{ and } \beta_6 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6.$$

The orbits containing simple roots look as follows:

$$\mathcal{O}_1 = \{\beta_1, \beta_2, \beta_3, \beta_4, -(\beta_1 + \beta_2 + \beta_3 + \beta_4)\}, \mathcal{O}_2 = \{\beta_6, -\beta_6\} \text{ and } \mathcal{O}_3 = \{\beta_5, \beta_2 + \beta_4 + \beta_5 + \beta_6, \beta_1 + \beta_3 + \beta_4 + \beta_5 + \beta_6, \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5, \beta_2 + \beta_3 + 2\beta_4 + \beta_5 + \beta_6, \beta_3 + \beta_4 + \beta_5, \beta_5 + \beta_6, \beta_1 + \beta_3 + \beta_4 + \beta_5, \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5 + \beta_6, \beta_2 + \beta_3 + 2\beta_4 + \beta_5\}.$$

The degrees are given by  $d_{\mathcal{O}_1} = d_{\mathcal{O}_2} = 0$  and  $d_{\mathcal{O}_3} = -1$ . Hence  $P(\check{b}) = P_{-\beta_5}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = 3$ .

**E<sub>7</sub>:** Consider  $\sigma = id$ . We have to find a new basis of the root system:

$$\beta_1 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7), \beta_2 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \beta_3 = \alpha_1, \\ \beta_4 = \alpha_2 + \alpha_3 + \alpha_7, \beta_5 = \alpha_3 + \alpha_4, \beta_6 = -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)$$

and  $\beta_6 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_7)$ .

Then, the cox-orbits involving the basis elements have the following form:

$$\begin{aligned}\mathcal{O}_1 &= \{\beta_1, \beta_2, -\beta_1 - \beta_2\}, \mathcal{O}_2 = \{\beta_4, \beta_5, \beta_6, -\beta_4 - \beta_5 - \beta_6\}, \mathcal{O}_3 = \{\beta_7, -\beta_7\} \text{ and} \\ \mathcal{O}_4 &= \{\beta_3, \beta_2 + \beta_3, \beta_2 + \beta_3 + \beta_4 + \beta_7, \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5, \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7, \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_7, \beta_3 + \beta_4 + \beta_5, \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7, \beta_1 + \beta_2 + \beta_3, \beta_3 + \beta_4 + \beta_7, \beta_2 + \beta_3 + \beta_4 + \beta_5, \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7\}.\end{aligned}$$

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = d_{\mathcal{O}_2} = d_{\mathcal{O}_3} = 0$  and  $d_{\mathcal{O}_4} = -1$ . Hence,  $P(\check{b}) = P_{-\beta_3}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = 8$ .

For  $\sigma = \gamma$  we get  $\check{\lambda} = \check{\lambda}_7$  and  $w$  interchanging  $\alpha_0$  with  $\alpha_7$ ,  $\alpha_1$  with  $\alpha_6$  and  $\alpha_3$  with  $\alpha_5$  fixing  $\alpha_2$  and  $\alpha_4$ . Thus  $\text{cox}^\sigma = s_7 s_6 s_5 s_4 s_2 \check{\lambda}_7 w$ . A basis in which the parabolic subgroup will be of standard type looks as follows:

$$\begin{aligned}\beta_1 &= \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \beta_2 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \beta_3 = \alpha_1 + \alpha_3, \beta_4 = \alpha_4 + \alpha_5, \\ \beta_5 &= -\alpha_5, \beta_6 = -(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) \text{ and } \beta_7 = -(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6).\end{aligned}$$

The orbits containing simple roots are:

$$\mathcal{O}_1 = \{\beta_1, \beta_2, \beta_3, \beta_4, -(\beta_1 + \beta_2 + \beta_3 + \beta_4)\}, \mathcal{O}_2 = \{\beta_6, \beta_7, -(\beta_6 + \beta_7)\} \text{ and } \mathcal{O}_3 = \{\beta_5, \beta_1 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7, \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5 + \beta_6, \beta_2 + \beta_3 + 2\beta_4 + \beta_5, \beta_2 + \beta_4 + \beta_5 + \beta_6 + \beta_7, \beta_5 + \beta_6, \beta_1 + \beta_3 + \beta_4 + \beta_5, \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5 + \beta_6 + \beta_7\}.$$

We calculate for the degrees  $d_{\mathcal{O}_1} = d_{\mathcal{O}_2} = 0$  and  $d_{\mathcal{O}_3} = 1$  implying  $P(\check{b}) = P_{\beta_5}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = 5$ .

**E<sub>8</sub>:** For  $\sigma = id$  a new basis of the root system is given by:

$$\begin{aligned}\beta_1 &= -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + \alpha_7 + 2\alpha_8), \beta_2 = \alpha_4 + \alpha_5 + \alpha_6, \\ \beta_3 &= \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8, \beta_4 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \beta_5 = \alpha_1, \beta_6 = \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_8, \\ \beta_7 &= -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8) \text{ and } \beta_7 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8).\end{aligned}$$

Then, the cox-orbits involving the basis elements have the following form:

$$\begin{aligned}\mathcal{O}_1 &= \{\beta_1, \beta_2, \beta_3, \beta_4, -\beta_1 - \beta_2 - \beta_3 - \beta_4\}, \mathcal{O}_2 = \{\beta_6, \beta_7, -\beta_6 - \beta_7\}, \mathcal{O}_3 = \{\beta_8, -\beta_8\} \text{ and} \\ \mathcal{O}_4 &= \{\beta_1 \dots \beta_5, \beta_1 \dots \beta_5 + \beta_6, \beta_1 \dots \beta_5 + \beta_6 + \beta_7, \beta_1 \dots \beta_5 + \beta_8, \beta_1 \dots \beta_5 + \beta_6 + \beta_8, \beta_1 \dots \beta_5 + \beta_6 + \beta_7 + \beta_8, 1 \leq i \leq 5\}.\end{aligned}$$

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = d_{\mathcal{O}_2} = d_{\mathcal{O}_3} = 0$  and  $d_{\mathcal{O}_4} = -1$ . Hence,  $P(\check{b}) = P_{-\beta_5}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = 9$ .

**F<sub>4</sub>:** For  $\sigma = id$  a new basis of the root system has to be introduced:

$$\beta_1 = -(\alpha_1 + \alpha_2 + 2\alpha_3), \beta_2 = \alpha_1, \beta_3 = \alpha_2 + \alpha_3, \beta_4 = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4).$$

Then, the cox-orbits involving the basis elements have the following form:

$$\begin{aligned}\mathcal{O}_1 &= \{\beta_1, -\beta_1\}, \mathcal{O}_2 = \{\beta_3, \beta_4, -\beta_3 - \beta_4\}, \mathcal{O}_3 = \{\beta_2, \beta_1 + \beta_2, \beta_2 + 2\beta_3, \beta_2 + \beta_3 + 2\beta_4, \beta_1 + \beta_2 + 2\beta_3\beta_1 + \beta_2 + 2\beta_3 + 2\beta_4\}.\end{aligned}$$

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = d_{\mathcal{O}_2} = 0$  and  $d_{\mathcal{O}_3} = -1$ . Hence,  $P(\check{b}) = P_{-\beta_2}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = 5$ .

**G<sub>2</sub>:** Consider  $\sigma = id$ . Introduce a new basis of the root system:

$$\beta_1 = \alpha_1, \beta_2 = -\alpha_1 - \alpha_2.$$

Then, the cox-orbits involving the basis elements have the following form:

$$\mathcal{O}_1 = \{\beta_2, -\beta_2\}, \mathcal{O}_3 = \{\beta_1, \beta_1 + 3\beta_2\}.$$

Evaluation on  $\check{b}$  yields:  $d_{\mathcal{O}_1} = 0$  and  $d_{\mathcal{O}_3} = -1$ . Hence,  $P(\check{b}) = P_{-\beta_1}$  and  $\dim \text{Aut}_G(\xi_\sigma)^+ = 3$ .

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